MTH 301: Group Theory

Assignment II: Group Actions

Practice assignment

1. Show that each of the following maps define an action. Furthermore, determine the faithfulness of the actions, characterize the orbits and stabilizers of the actions, and verify the orbit-stabilizer theorem wherever applicable.

   (a) For a set $X \neq \emptyset$, $S(X) \times X \to X : (f, x) \mapsto f(x)$.

   (b) For a group G, $\text{Aut}(G) \times G \to G : (\varphi, g) \mapsto \varphi(g)$.

   (c) $S_n \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} : (\sigma, i) \mapsto \sigma(i)$.

   (d) $D_{2n} \times \{e^{i2\pi k/n} : 0 \leq k \leq n-1\} \to \{e^{i2\pi k/n} : 0 \leq k \leq n-1\} :$ $(r, e^{i2\pi k/n}) \mapsto e^{i2\pi(k+1)/n}$ and $(s, e^{i2\pi k/n}) \mapsto e^{-i2\pi k/n}$.

   (e) $\mathbb{R} \times \mathbb{C} \to \mathbb{C} : (x, re^{i\theta}) \mapsto re^{i(\theta+x)}$.

   (f) $\mathbb{Z}_2 \times S^2 \to S^2 : (1, (x, y, z)) \mapsto (-x, -y, -z)$, where S^2 is unit sphere centered at origin in $\mathbb{R}^3$.

   (g) $\text{GL}(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n : (A, v) \mapsto Av$.

2. Establish the assertion in and 4.4 (xi) of the Lesson Plan.

3. Show that a normal subgroup is a disjoint union of conjugacy classes including the trivial conjugacy class.

4. Let G be a group, $H < G$, and $S \subset g$.

   (a) Let $\langle\langle S \rangle\rangle$ be the intersection of all normal subgroups of G containing S (also known as the *normal closure* of S in G). Show that
   $$\langle\langle S \rangle\rangle = \langle \{gsg^{-1} : g \in G \text{ and } s \in S\} \rangle.$$
   (Note that $\langle\langle S \rangle\rangle$ is also the smallest normal subgroup of G containing S.)

   (b) Show that $H \lhd N_G(H)$. Furthermore, show that $N_G(H)$ is the largest subgroup of G in which H is normal.

(c) Show that
$$Z(G) = \bigcap_{g \in G} C_G(g).$$

(d) Show that if all elements of S commute with each other, then the largest subgroup of G whose center contains S is $C_G(S)$.

(e) Show that $C_G(S) \lhd N_G(S)$. Moreover,
$$N_G(H)/C_G(H) \cong K,$$
where $K < \mathrm{Aut}(H)$.

5. Consider the group A_n for $n \geq 3$.
   (a) Classify the normal subgroups of A_4.
   (b) Compute the conjugacy classes of A_5.
   (c) Show that A_n is generated by the set of 3-cycles $\{(a\,b\,c) : 1 \leq a < b < c \leq n\}$.
   (d) For $n \geq 5$, show that any two 3-cycles in A_n are conjugate.

Problems for submission

(Due: 14/09/2023)

1. Establish the assertions in 4.3.2 (iii) of the Lesson Plan.
2. Solve 1(c) and 1(d) from the practice assignment above. Use your solutions to show that $D_{2n} < S_n$ for $n \geq 3$.